

KNOTTED SURFACES IN 4-MANIFOLDS BY KNOT SURGERY AND STABILIZATION

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ABSTRACT. Given a simply-connected closed 4-manifold X and a smoothly embedded oriented surface Σ , various constructions based on Fintushel-Stern knot surgery have produced new surfaces in X that are pairwise homeomorphic to Σ , but not diffeomorphic. We prove that for all known examples of surface knots constructed from knot surgery operations that preserve the fundamental group of the complement of surface knots, they become pairwise diffeomorphic after stabilizing by connected summing with one $S^2 \tilde{\times} S^2$. When X is spin, we show in addition that any surfaces obtained by a knot surgery whose complements have cyclic fundamental group become pairwise diffeomorphic after one stabilization by $S^2 \tilde{\times} S^2$.

1. INTRODUCTION

Let X be a smooth closed 4-manifold and Σ be a smoothly embedded surface. An ‘exotic embedding’ of a surface Σ in X is a smooth embedding in X that is pairwise homeomorphic to Σ , but not diffeomorphic. The ‘stabilization’ of given pair (X, Σ) is the process of connected summing with a standard manifold pair $(S^2 \times S^2, \emptyset)$ or $(S^2 \tilde{\times} S^2, \emptyset)$, where $S^2 \tilde{\times} S^2$ denotes the non-trivial S^2 bundle over S^2 . The recent work [3] of Auckly, Melvin, Ruberman, and the author has constructed the first examples of exotic 2-spheres in closed simply-connected 4-manifolds that become pairwise smoothly isotopic after ‘single’ stabilization by $(S^2 \times S^2, \emptyset)$. In this context, one can ask if this stabilization phenomenon arises to exotic surfaces with higher genus.

While a great deal of exotic embeddings in 4-manifolds are known through various constructions [7, 6, 9, 14, 15, 16, 17], interestingly most examples of exotic embeddings for oriented surfaces in simply-connected 4-manifolds derive from the constructions based on ‘knot surgery’ of Fintushel-Stern [8]. Knot surgery using a knot K in S^3 is the operation of removing a neighborhood of a torus T and replacing it by a product of S^1 and the exterior of the knot K . Fintushel and Stern provided an effective way to detect the change of diffeomorphism type for knot surgery, showing that the Alexander polynomial of K is reflected in the Seiberg-Witten invariant for a knot surgered 4-manifold. This allows one to quickly construct and detect infinite families of exotic smooth structures on a large class of 4-manifolds. Likewise, knot surgery can be used to change a smooth structure of smoothly embedded surface in a 4-manifold. This approach relies on ‘ambient surgery’ whereby a given surface Σ is surgered to a new surface $\Sigma_K(\varphi)$, leaving the ambient manifold X fixed. The rim surgery of Fintushel-Stern [9], author’s twist rim surgery [14], and Finashin’s annulus rim surgery [6] are examples of this technique, underlying most examples of smoothly knotted oriented surfaces in a simply-connected closed 4-manifold.

In the direction of the study of stabilization for exotic smooth structures, Auckly [2] for $S^2 \tilde{\times} S^2$ and Akbulut [1] for $S^2 \times S^2$ proved that a simply-connected 4-manifold X and its knot surgered manifold $X_K(\varphi)$ become diffeomorphic after single stabilization by $S^2 \tilde{\times} S^2$

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or $S^2 \times S^2$, referred to as *1-stably equivalent* with the terminology in [3]; see [4] for the alternative proof.

This paper investigates the analogous stabilization question for knotted surfaces produced by all of the known constructions based on knot surgery i.e. rim surgery, twist rim surgery, and annulus rim surgery.

The Wall's stable h -cobordism theorem [27] states that homotopy equivalent, simply-connected 4-manifolds become diffeomorphic after stabilization by some finite number of $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$. It also holds for embedded surfaces (up to diffeomorphism of pairs) with simply connected complements in a 4-manifold that represent the same homology class [21]. And, in fact, all known examples need only one stabilization to be diffeomorphic. So, the stabilization question for a knot surgered pair $(X, \Sigma_K(\varphi))$ would be the following. In this paper, we will use the terminology 'surface knot group' for the fundamental group of surface complement in a 4-manifold.

Question 1.1. Suppose that X is simply connected and Σ is an oriented smoothly embedded surface. Let $(X, \Sigma_K(\varphi))$ be a pair obtained by a knot surgery from (X, Σ) . If $\Sigma_K(\varphi)$ and Σ have the same surface knot group in X then are they 1-stably equivalent?

This paper answers this question for all of the currently known constructions. The precise statements are given in Section 2 (Theorems A, B, C) after we discuss the known techniques for constructing exotic surfaces.

Rim surgery of Fintushel and Stern [9] constructed an infinite family of exotic smooth embedding for surfaces with simply-connected complements in a simply-connected 4-manifold. Finashin used annulus rim surgery [6] for knotting algebraic curves in \mathbb{CP}^2 , and produced surfaces that are smoothly not isotopic to algebraic curves for degree $d \geq 5$, but the topological classification of his examples was open. The later work [14] of the author introduced a method, called twist rim surgery, of knotting surfaces that produced exotic embeddings for surfaces with cyclic knot groups in a simply-connected 4-manifold. Applied to algebraic curves in \mathbb{CP}^2 , the twist rim surgery leads to the construction of infinitely many exotic smooth structures on algebraic curves of degree $d \geq 3$. For degrees 1 and 2, the surfaces are spheres, and it is not easy to distinguish these by Seiberg-Witten invariants. The work of Ruberman and author [15] strengthened the criterion from [14] for topological equivalence of surfaces by showing that any surfaces produced by a knot surgery that preserve a cyclic knot group is topologically standard. As a consequence, we deduced that Finashin's examples are topologically standard. Despite some results about the existence of symplectic, noncomplex surfaces as well as smooth surfaces without symplectic structures, the main classical source of examples for smooth embeddings codimension 2 had been complex curves. A subsequent work [16] extended Gompf's theorem about the fundamental group of symplectic manifolds to the relative case, showing that any finitely presented group can be realized as the fundamental group of complement of a symplectic surface in a simply-connected symplectic 4-manifold, whereas the fundamental groups of complement of complex curves are quite restricted. Those examples can be further smoothly knotted by twist rim surgery so that it has led to a large class of exotic embeddings. Another interesting aspect of twist rim surgery is that some iteration of the twist rim surgery gives a way of constructing new smooth surfaces with certain non-abelian finite surface knot group. One consequence is that it gave an infinite family of exotic surfaces in $S^2 \times S^2$ with knot group a dihedral group D_{2p} , for any odd p .

In this paper, we prove that for all known examples of surface knots constructed from rim surgery, twisted rim surgery, and annulus rim surgery that preserve their surface knot groups, they become pairwise diffeomorphic after ‘single’ stabilization by $(S^2 \tilde{\times} S^2, \emptyset)$.

Another result includes an interesting phenomenon in the relative version of stabilization i.e. connected sum with $(S^2 \times S^2, \emptyset)$ or $(S^2 \tilde{\times} S^2, \emptyset)$. It is known that for a nonspin simply-connected 4-manifold X , essentially due to Wall [26], $X \# S^2 \times S^2$ is diffeomorphic to $X \# S^2 \tilde{\times} S^2$, but surprisingly it is not true for the relative case. Theorem D proves that for a degree d -curve Σ_d in \mathbb{CP}^2 , $(\mathbb{CP}^2 \# S^2 \times S^2, \Sigma_d)$ is *not* even pairwise homeomorphic to $(\mathbb{CP}^2 \# S^2 \tilde{\times} S^2, \Sigma_d)$, even when $\mathbb{CP}^2 - \Sigma_d$ is not spin i.e. $d = \text{even}$.

Finally, we show that if a knot surgery $(X, \Sigma) \rightarrow (X_K(\varphi), \Sigma_K(\varphi))$ is cyclic, which is defined to be a surgery preserving a cyclic surface knot group [15], then the pairs are 1-stably equivalent by connected summing with $(S^2 \tilde{\times} S^2, \emptyset)$ in the case that X is spin.

Remark 1.2. Note that here we will not impose any extra assumptions on Σ other than that Σ is an oriented smoothly embedded surface in a simply-connected closed 4-manifold X . Recall that the constructions of (twisted) rim surgery and annulus rim surgery can provide exotic embeddings of Σ when Σ is a surface of positive genus and (X, Σ) has a non-trivial relative Seiberg-Witten invariant [8, 9, 10, 25] (or a relative Heegaard-Floer invariant as in the version of Mark [19]).

The main theorems are precisely stated in the next section where it carefully describes when surface knot groups are preserved for each knotting construction. And it includes the proof of Theorem D.

2. MAIN THEOREMS

Before we state our results, the notions of ‘equivalence’ of embeddings of surfaces in a 4-manifold should be clarified as in [3]:

Definition 2.1. Two smoothly embedded surfaces Σ, Σ' in a smooth 4-manifold Z are *equivalent* if there is an orientation preserving pairwise diffeomorphism of (Z, Σ) to (Z, Σ') . Two smoothly embedded surfaces Σ, Σ' in a smooth 4-manifold X are *n -stably equivalent* if the natural embeddings $\Sigma, \Sigma' \subset X \# nS^2 \times S^2$ (or $nS^2 \tilde{\times} S^2$) are equivalent in $X \# nS^2 \times S^2$ (or $nS^2 \tilde{\times} S^2$), but not in $X \# kS^2 \times S^2$ (or $kS^2 \tilde{\times} S^2$) for any $k \leq n - 1$.

Note that our constructed exotic 2-spheres in [3] have simply-connected complements and they are 1-stably *isotopic* which is a stronger notion of equivalence of surfaces. It is still open to see the distinction between equivalence of surfaces up to *diffeomorphism* and *smooth isotopy* [23, 24], while this issue does not arise in the topological case [20, 22]. Here our stabilization by $S^2 \times S^2$ (or $S^2 \tilde{\times} S^2$) is taken in the ‘outside’ of embedded surfaces in X , but there is another notion of stabilization for embedded surfaces, adding an unknotted handle to the surface. The recent work [5] of Baykur-Sunukjian showed that all constructions of exotic knotting of surfaces produce surfaces that become smoothly *isotopic* after adding a single handle in a standard way.

Let X be a smooth 4-manifold containing a torus T with a trivial normal bundle and let K be a knot in S^3 with its closed complement $E(K)$. Fintushel-Stern’s knot surgery [8] is the process of removing a neighborhood of T from X and re-gluing $S^1 \times E(K)$ via a diffeomorphism φ on the boundary to form $X_K(\varphi) = X - \nu(T) \cup_{\varphi} S^1 \times E(K)$. Denote by μ_T the boundary of the normal disk of T , and let the meridian/longitude of K be μ_K and λ_K respectively. Here the gluing map $\varphi : \partial\nu(T) \rightarrow S^1 \times \partial E(K)$ can be chosen

by any diffeomorphism such that $\varphi_*\mu_T = \lambda_K$. When X is a simply-connected closed 4-manifold, this operation doesn't change the homeomorphism type, while it may change its diffeomorphism type.

Applied to a torus in the exterior of an embedded surface in a closed 4-manifold, the knot surgery can change embeddings of surfaces in 4-manifolds. We assume that X is a smooth simply-connected closed 4-manifold, and Σ is an oriented embedded surface in X throughout the paper. Then the fundamental group $\pi_1(X - \Sigma)$ is normally generated by a meridian μ_Σ of surface. For a surface Σ carrying a non-trivial homology class in X , the first homology group $H_1(X - \Sigma)$ is always finite cyclic, of order that we will usually write as d . The process of knotting an embedded surface Σ in X can be obtained by performing knot surgery on a torus in the exterior $X - \nu(\Sigma)$, that usually links with Σ in a simply way, and then gluing back in the neighborhood of the surface gives a new embedding of Σ in $X_K(\varphi)$ with image $\Sigma_K(\varphi)$. In the case of rim surgery, twist rim surgery, and annulus rim surgery, there is a canonical identification between X and $X_K(\varphi)$ so that we can view $\Sigma_K(\varphi)$ as an embedding in X ; see Section 3 for more details of these constructions. In general the resulting homeomorphism/diffeomorphism type of the new embedding $\Sigma_K(\varphi)$ depends on a choice of torus T , knot K , and gluing map φ . Our results show that the surfaces $\Sigma_K(\varphi)$ and Σ are 1-stably equivalent under some circumstances as follows.

Rim surgery deals with surfaces with simply-connected complements in a simply-connected 4-manifold and doesn't change the fundamental group, so the surface $\Sigma_K(\varphi)$ is in fact topologically isotopic to Σ by the works in [20, 22]. The following theorem shows the stabilization result for these surfaces.

Theorem A. *Suppose that X is a simply-connected closed 4-manifold and Σ is an smoothly embedded oriented surface with $\pi_1(X - \Sigma) = 1$. Let $(X, \Sigma_K(\varphi))$ be a pair obtained by a rim surgery. Then $(X, \Sigma) \# (S^2 \tilde{\times} S^2, \emptyset)$ is pairwise diffeomorphic to $(X, \Sigma_K(\varphi)) \# (S^2 \tilde{\times} S^2, \emptyset)$.*

Finashin's annulus rim surgery [6] requires a suitable annulus $M \cong S^1 \times I$ in X to produce a new surface via knotting Σ along the annulus. This surgery in his paper is given by an explicit geometric description of the surgered surface, but in [15] a knot surgery description for this surgery is provided; see Section 3.3 for this description. It is shown in [6, 15] that annulus rim surgery preserves the surface knot group when $\pi_1(X - \Sigma) = \mathbb{Z}_d$, and it turns out that the surface $\Sigma_K(\varphi)$ is topologically isotopic to Σ by the work in [15, Theorem 1.3].

Theorem B. *Suppose that X is a simply-connected closed 4-manifold and Σ is an smoothly embedded oriented surface with $\pi_1(X - \Sigma) = \mathbb{Z}_d$. Let $(X, \Sigma_K(\varphi))$ be a pair obtained by an annulus rim surgery. Then $(X, \Sigma) \# (S^2 \tilde{\times} S^2, \emptyset)$ is pairwise diffeomorphic to $(X, \Sigma_K(\varphi)) \# (S^2 \tilde{\times} S^2, \emptyset)$.*

In order to explore this phenomenon for surface knots with arbitrary knot groups, we consider twist rim surgery [14, 15, 16], a variation of the Fintushel-Stern's rim surgery with additional twists parallel to a meridian and a longitude of a knot K . We write the meridian twist rim surgery as ' m -twist rim surgery' when we wish to indicate the number of twists applied on the meridian of K , and also denote by $\Sigma_K(m)$ the new embedding produced from Σ under the surgery. The way in which m -twist rim surgery affects the fundamental group of a surface knot depends to some degree on the relation between m and d , where $H_1(X - \Sigma) \cong \mathbb{Z}_d$. For example, when $m = \pm 1$, the twist rim surgery always preserves the fundamental group of a surface knot; the proof was given for 1-twist in [16, Proposition 2.3], but it works for -1 -twist in the exactly same way. The 1-twist rim surgery allows us to construct exotic smooth embeddings for a symplectic surface with

any finitely presented knot group in a symplectic 4-manifold (see [16, Theorem 3.1, 5.2] for more details). More generally, Proposition 2.4 in [16] shows when an m -twist rim surgery preserves the fundamental group of surface knots as shown that for a surface $\Sigma \subset X$ with $H_1(X - \Sigma) \cong \mathbb{Z}_d$, if $(m, d) = 1$ and the meridian μ_Σ has order d in $\pi_1(X - \Sigma)$, then $\pi_1(X - \Sigma) \cong \pi_1(X - \Sigma_K(m))$. This criterion is used to produce infinitely many exotic embeddings in $S^2 \times S^2$ with knot group a dihedral group D_{2p} for any odd p [16, Theorem 5.1]. Note that when $\pi_1(X - \Sigma) = \mathbb{Z}_d$, the surface $\Sigma_K(m)$ with $(m, d) = 1$ is topologically isotopic to Σ ; see [15, Theorem 1.3]. But for arbitrary surface knot groups, when the knot K is chosen carefully, $\Sigma_K(m)$ is equivalent to Σ up to smooth s -cobordism; see [16] for more details. In all cases that surface knot groups under twist rim surgery are preserved, we show that $\Sigma_K(m)$ and Σ are 1-stably equivalent:

Theorem C. *Suppose that the surface $\Sigma \subset X$ has $H_1(X - \Sigma) \cong \mathbb{Z}_d$, and let $\pi_1(X - \Sigma)$ be any group G . Then the following is true.*

- (1) $(X \# S^2 \tilde{\times} S^2, \Sigma)$ is pairwise diffeomorphic to $(X \# S^2 \tilde{\times} S^2, \Sigma_K(\pm 1))$.
- (2) If $(m, d) = 1$ and μ_Σ has order d in $\pi_1(X - \Sigma)$ then $(X \# S^2 \tilde{\times} S^2, \Sigma)$ is pairwise diffeomorphic to $(X \# S^2 \tilde{\times} S^2, \Sigma_K(m))$.

Now, we give a simple proof to show an interesting phenomenon in this relative stabilization. Wall's stabilization result [26] for a nonspin simply-connected 4-manifold X shows that $X \# S^2 \times S^2$ is diffeomorphic to $X \# S^2 \tilde{\times} S^2$, but interestingly it fails as follows:

Theorem D. *Let Σ_d be a degree d -curve in \mathbb{CP}^2 . Then the pair $(\mathbb{CP}^2 \# S^2 \times S^2, \Sigma_d)$ is 'not' pairwise homeomorphic to $(\mathbb{CP}^2 \# S^2 \tilde{\times} S^2, \Sigma_d)$.*

Proof. If d is odd then it is obvious since $\mathbb{CP}^2 - \Sigma_d$ is spin. But, we will show that the pairs are still not homeomorphic in the case that d is even so that $\mathbb{CP}^2 - \Sigma_d$ is not spin. We claim that there is no immersed sphere $S \looparrowright (\mathbb{CP}^2 - \Sigma_d) \# S^2 \times S^2$ such that its self intersection number $[S]^2$ is odd. The Hopf sequence shows

$$\pi_2((\mathbb{CP}^2 - \Sigma_d) \# S^2 \times S^2) \xrightarrow{h} H_2((\mathbb{CP}^2 - \Sigma_d) \# S^2 \times S^2) \rightarrow H_2(\pi_1((\mathbb{CP}^2 - \Sigma_d) \# S^2 \times S^2)) \rightarrow 0.$$

Since $\pi_1((\mathbb{CP}^2 - \Sigma_d) \# S^2 \times S^2) = \mathbb{Z}_d$ and $H_2(\mathbb{Z}_d) = 0$, the Hurewicz map h is onto, so every element of $H_2((\mathbb{CP}^2 - \Sigma_d) \# S^2 \times S^2)$ is represented by an immersed sphere. Let's assume that there exists an immersed sphere S in $(\mathbb{CP}^2 - \Sigma_d) \# S^2 \times S^2$ with odd intersection number. Then the homology class $[S]$ can be written by $k[\mathbb{CP}^1] + n_a S_a + n_b S_b$ in $H_2(\mathbb{CP}^2 \# S^2 \times S^2)$, where S_a and S_b denote the first and second generators of $H_2(S^2 \times S^2)$ respectively. It gives $[S] \cdot [\Sigma_d] = kd$ that must be zero, so $k = 0$. This implies that there is no odd class in $H_2((\mathbb{CP}^2 - \Sigma_d) \# S^2 \times S^2)$, but there is in $H_2((\mathbb{CP}^2 - \Sigma_d) \# S^2 \tilde{\times} S^2)$. \square

Remark 2.2. It is worth pointing out that there is no odd class in $\mathbb{CP}^2 - \Sigma_d$ even when $\mathbb{CP}^2 - \Sigma_d$ is not spin. To understand this, first note that the nonzero element $\alpha \in H_2(\mathbb{CP}^2 - \Sigma_d)$ has $\alpha^2 = 0$ as shown in the above proof, so it gives $Q_{\mathbb{CP}^2 - \Sigma_d} = 0$. In fact the handlebody picture of $\mathbb{CP}^2 - \Sigma_d$ shows that there are $2g$ 0-framed 2-handles, and a 1-framed 2-handle which is d -times linked with a 1-handle. For $d = \text{even}$, there is a \mathbb{Z}_2 -homology class β of the 1-framed 2-handle and over \mathbb{Z}_2 , the intersection form is given by [1]. By the Wu formula, $w_2(\mathbb{CP}^2 - \Sigma_d)$ vanishes on α , but has value 1 on the \mathbb{Z}_2 -homology class β .

Finally, we focus on the case that $\pi_1(X - \Sigma)$ is a cyclic group \mathbb{Z}_d , and investigate the stabilization problem of knot surgery. As the terminology in [15], if a knot surgery $(X, \Sigma) \rightarrow (X_K(\varphi), \Sigma_K(\varphi))$ satisfies that $\pi_1(X - \Sigma) \cong \pi_1(X_K(\varphi) - \Sigma_K(\varphi))$ is cyclic then the knot

surgery is called a *cyclic surgery*. In [15, Theorem 1.2], Ruberman and the author showed that for any pair (X, Σ) that X is simply-connected and Σ is an embedded surface with $\pi_1(X - \Sigma) \cong \mathbb{Z}_d$, if a knot surgery $(X, \Sigma) \rightarrow (X_K(\varphi), \Sigma_K(\varphi))$ is cyclic then there is a pairwise homeomorphism $(X, \Sigma) \rightarrow (X_K(\varphi), \Sigma_K(\varphi))$. Thus, it is natural to ask the 1-stable equivalence for the cyclic knot surgery. We answer for this question in the case that X is spin:

Theorem E. *Let X be a simply-connected, closed, spin 4-manifold and Σ be an embedded oriented surface with $\pi_1(X - \Sigma) \cong \mathbb{Z}_d$. Suppose that the knot surgery $(X, \Sigma) \rightarrow (X_K(\varphi), \Sigma_K(\varphi))$ is cyclic. Then (X, Σ) is pairwise diffeomorphic to $(X_K(\varphi), \Sigma_K(\varphi))$ after one stabilization with $(S^2 \tilde{\times} S^2, \emptyset)$.*

Remark 2.3. In the contrast to the well-known stabilization theorems for simply-connected 4-manifolds, the relative stabilization for cyclic knot surgery doesn't seem to give any general statement for a choice of $(S^2 \times S^2, \emptyset)$ or $(S^2 \tilde{\times} S^2, \emptyset)$ in the case of a nonspin 4-manifold X . Our main argument for stabilization results will rely on proving the 1-stable equivalence of surface knots $\Sigma_{K_i}(\varphi)$ and $\Sigma_{K_{i+1}}(\varphi)$ for two knots K_i and K_{i+1} related by one crossing change. At each stage of crossing change that will make any knot to an unknot, one cannot assert that the pairs $(X_{K_i}(\varphi), \Sigma_{K_i}(\varphi))$ and $(X_{K_{i+1}}(\varphi), \Sigma_{K_{i+1}}(\varphi))$ become pairwise diffeomorphic after one stabilization with *only* $(S^2 \times S^2, \emptyset)$ or with *only* $(S^2 \tilde{\times} S^2, \emptyset)$ when $X - \Sigma$ is nonspin. This issue arises because $(X_{K_i}(\varphi), \Sigma_{K_i}(\varphi)) \# (S^2 \times S^2, \emptyset)$ may not be pairwise diffeomorphic to $(X_{K_i}(\varphi), \Sigma_{K_i}(\varphi)) \# (S^2 \tilde{\times} S^2, \emptyset)$ as seen in the proof of Theorem D.

3. KNOT SURGERY CONSTRUCTIONS TO CHANGE EMBEDDINGS IN 4-MANIFOLDS

Let X be a simply-connected closed 4-manifold and Σ be an embedded oriented surface.

3.1. Twist rim surgery. Let R_α be a torus with $R_\alpha \cdot R_\alpha = 0$ (called a *rim torus*) that is the preimage in $\partial\nu(\Sigma)$ of a closed curve $\alpha \subset \Sigma$. Identify the neighborhood $\nu(\alpha)$ of the curve α in X with $S^1 \times I \times D^2 = S^1 \times B^3$ where $\nu(\alpha)$ in Σ is $S^1 \times I$. In this trivialization, let β be a pushed-in copy of the meridian circle $\{0\} \times \partial D^2 \subset I \times D^2$, so it is isotopic to a meridian of Σ . Then the rim torus R_α can be written as $\alpha \times \beta \subset S^1 \times (B^3, I)$ and we will identify a neighborhood $\nu(R_\alpha)$ of R_α with $\alpha \times (\beta \times D^2) \subset S^1 \times (B^3, I)$. Let K be a knot in S^3 with its exterior $E(K)$, and μ_K, λ_K denotes a pair of meridian-longitude of K . The m -twists and n -rolls of rim surgery on (X, Σ) is defined by

$$(X, \Sigma_K(\varphi)) = (X, \Sigma) - \nu(R_\alpha) \cup_\varphi S^1 \times E(K).$$

Here the gluing map $\varphi : \partial\nu(R_\alpha) \rightarrow S^1 \times \partial E(K)$ is the diffeomorphism determined by

$$(1) \quad \varphi_*(\alpha') = m\mu_K + n\lambda_K + [S^1], \quad \varphi_*(\beta') = \mu_K, \quad \text{and} \quad \varphi_*(\mu_R) = \lambda_K$$

with respect to a basis $\{\alpha', \beta', \mu_R\}$ for $H_1(\partial\nu(R_\alpha))$ and $\{[S^1], \mu_K, \lambda_K\}$ for $H_1(S^1 \times \partial E(K))$, where α', β' are the pushoffs of α, β into $\partial\nu(R_\alpha)$ and μ_R denotes a meridian of the rim torus.

Such a gluing corresponds to the *spinning* construction of the rim surgery of Fintushel-Stern i.e. $m = n = 0$, adding a combination of m -fold twist spinning [28] and n -fold roll spinning [11, 18]. It is useful to specify these twists by classical diffeomorphisms that give equivalent descriptions for the twisted rim surgery.

Consider self-diffeomorphisms denoted by τ and ρ of (S^3, K) that correspond to twists parallel to a meridian and a longitude of K respectively. Let $\partial E(K) \times I = K \times \partial D^2 \times I$ be

a collar of $\partial E(K)$ in $E(K)$ under a suitable trivialization with 0-framing. Identify K with $S^1 \cong \mathbb{R}/\mathbb{Z}$ and then the twist map τ is given by

$$(2) \quad \tau(\bar{\theta}, e^{i\psi}, t) = (\bar{\theta}, e^{i(\psi+2\pi t)}, t) \quad \text{for} \quad (\bar{\theta}, e^{i\psi}, t) \in K \times \partial D^2 \times I$$

and otherwise, $\tau(y) = y$.

Similarly, a roll, ρ , is obtained from $\rho(\bar{\theta}, e^{i\psi}, t) = (\overline{\bar{\theta} + t}, e^{i\psi}, t)$ by extending as the identity on the rest of (S^3, K) . Although a roll can also produce exotic embeddings, we will only deal with an m -twist rim surgery in this paper since a meridian twist is sufficiently useful to construct all desired smoothly knotted surfaces. Most of the arguments for the stabilization result of the m -twist rim surgery be easily modified to address the rolling as well.

Writing $(S^3, K) = (B^3, K_+) \cup (B^3, K_-)$ where (B^3, K_-) is an unknotted ball pair, we regard τ as an automorphism of (B^3, K_+) . Since the rim torus R_α lies in a neighborhood of the curve α , the twisted rim surgery performed in $\nu(\alpha) \cong S^1 \times (B^3, I)$ gives rise to the mapping torus of (B^3, K_+) with monodromy given by the twist map τ . So the m -twisted rim surgery on (X, Σ) can be written as follows;

$$(3) \quad (X, \Sigma_K(m)) = (X, \Sigma) - S^1 \times (B^3, I) \cup_{\partial} S^1 \times_{\tau^m} (B^3, K_+).$$

In doing any rim surgery (twisted or otherwise) we assume that $\alpha \subset \Sigma$ is a curve for which there is a framing of $\nu(\Sigma)$ along α such that the pushoff of α into $\partial\nu(\Sigma)$ is null-homotopic in $X - \Sigma$. But we don't assume that α is a non-separating curve on Σ , which is necessary to distinguish the diffeomorphism type of $\Sigma_K(m)$ from that of Σ with Seiberg-Witten invariant. Note from [14, Lemma 2.2] that if α bounds a disk in Σ , the surface $\Sigma_K(m)$ is the connected sum of Σ with the m -twist spun knot $K(m)$ of Zeeman [28]. Our stabilization results include this example as well.

3.2. Twisted rim surgery and the surface knot group. As mentioned in Section 2, ± 1 -twist rim surgery always preserves surface knot groups [16, Proposition 2.3], and also Proposition 2.4 in [16] shows when an m -twist rim surgery preserves the fundamental group. Here we will revisit Proposition 2.4 with more elementary argument (compare the proof in [16]) since it explicitly provides the presentation of the fundamental group for later use.

Proposition 3.1 (Proposition 2.4 in [16]). *Let $\pi_1(X - \Sigma)$ be any group G . Suppose that the surface $\Sigma \subset X$ has $H_1(X - \Sigma) = \mathbb{Z}/d$ and the meridian μ_Σ has order d in $\pi_1(X - \Sigma)$. If $(m, d) = 1$ then $\pi_1(X - \Sigma_K(m))$ is isomorphic to G .*

Proof. In order to investigate a presentation of $\pi_1(X - \Sigma_K(m))$, we first consider the decomposition of $X - \Sigma_K(m)$ induced from (3):

$$(4) \quad X - \Sigma_K(m) = X - \Sigma - S^1 \times (B^3, I) \cup_{\partial} S^1 \times_{\tau^m} (B^3 - K_+).$$

Choosing a base point $*$ at the intersection of two components in this decomposition (4), we get the following diagram from the van Kampen theorem;

$$(5) \quad \begin{array}{ccc} & \pi_1(X - \Sigma - S^1 \times (B^3, I)) & \\ i_1 \nearrow & & \searrow j_1 \\ \pi_1(S^1 \times (\partial B^3 - \{\text{two points}\})) & & \pi_1(X - \Sigma_K(m)) \\ i_2 \searrow & & \nearrow j_2 \\ & \pi_1(S^1 \times_{\tau^m} (B^3 - K_+)) & \end{array}$$

In the diagram, each map is obviously induced by an inclusion and $\pi_1(S^1 \times (\partial B^3 - \{\text{two points}\}))$ is generated by two elements $[S^1]$ and μ . So, the relations in a presentation of $\pi_1(X - \Sigma_K(m))$ are given by $i_1[S^1] = \alpha'$ which is trivial by the assumption that the pushoff α' of α is null-homotopic in $X - \Sigma$, and $i_1(\mu)$ is a meridian μ_Σ of Σ in $\pi_1(X - \Sigma - S^1 \times (B^3, I)) \cong \pi_1(X - \Sigma)$. So, it leads the presentation for $\pi_1(X - \Sigma_K(m))$ as follows:

$$(6) \quad \langle \pi_1(X - \Sigma) * \pi_1(S^1 \times_{\tau^m} (B^3 - K_+)) \mid 1 = \delta, \mu_\Sigma = \mu_K \rangle,$$

where $\delta = [S^1]$, $\mu_K = [\mu_K]$ denote generators of $\pi_1(S^1 \times_{\tau^m} (B^3 - K_+), *)$ in Figure 1. Associated with the relations, this presentation becomes the following:

$$(7) \quad \langle \pi_1(X - \Sigma) * \pi_1(B^3 - K_+) \mid \mu_\Sigma = \mu_K, \mu_K^{-m} g \mu_K^m = g, \forall g \in \pi_1(B^3 - K_+) \rangle.$$

If $(m, d) = 1$ and $\mu_\Sigma^d = 1$, it obviously gives $\pi_1(X - \Sigma)$.

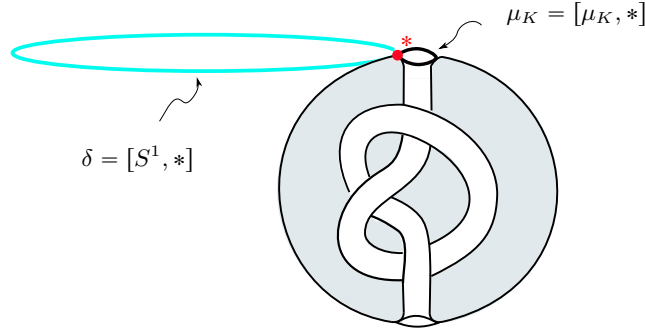


FIGURE 1. Generators of $\pi_1(S^1 \times_{\tau^m} (B^3 - K_+), *)$

□

Remark 3.2. Note that in the diagram, if the image of $\pi_1(S^1 \times_{\tau^m} (B^3 - K_+))$ in $\pi_1(X - \Sigma_K(m))$ is a cyclic subgroup generated by the meridian of $\Sigma_K(m)$, then the presentation (6) for $\pi_1(X - \Sigma_K(m))$ readily leads to the group $\pi_1(X - \Sigma)$. The choice of the parameter m in Proposition 3.1 makes this case. This property enables one to see that the stabilization result of Baykur-Sunukjian in [5] can be extended to knotted surfaces produced by the m -twist rim surgery with the choice of m in Proposition 3.1. They showed that for $m = 1$ or any m with $(m, d) = 1$ in the case of $\pi_1(X - \Sigma) = \mathbb{Z}_d$, the m -twist surgered surfaces become smoothly isotopic by adding one trivial handle. The same argument in [5, Section 3.2] can work for the case of Proposition 3.1 by adding a handle at a crossing of the knotted arc K_+ in $S^1 \times_{\tau^m} (B^3, K_+)$ to unknot it crossing by crossing, where it must be checked that the attached handle is trivial at each stage. It follows from that $\pi_1(S^1 \times_{\tau^m} (B^3 - K_+))$ is a cyclic subgroup generated by the meridian of $\Sigma_K(m)$; see [5, Lemma 3] for more details.

3.3. Annulus rim surgery. Suppose that there is a smoothly embedded annulus $M(\cong S^1 \times I$, where I denotes an interval $[-1, 1]$) in X such that M meets Σ normally along ∂M so that $M \cap \Sigma = \partial M$ are two curves α_{-1} and α_1 on Σ . We assume that $\Sigma - \{\alpha_{-1}, \alpha_1\}$ is connected. Choose a trivialization $\nu(M) \rightarrow (S^1 \times I) \times D^2 \cong S^1 \times B^3$ such that $M \cong (S^1 \times I) \times \{0\}$ and $\nu(M)|_\Sigma \cong S^1 \times f$, where f denotes a disjoint union of two unknotted segments $\partial I \times I \subset I \times D^2 = B^3$, a part of the boundary of a trivially embedded band $b = I \times I$ in B^3 (See Figure 2). So, M is identified with $S^1 \times I \times \{0\}$ in $S^1 \times b \subset S^1 \times B^3$.

Denote by m_b a meridian of b in B^3 and let T be a torus in $\nu(M)$ corresponding to $S^1 \times m_b \subset S^1 \times (B^3, f)$. Knot surgery along this torus T produces a new surface $\Sigma_K(\varphi)$.

The simplest gluing $\varphi : \partial\nu(T) \rightarrow S^1 \times \partial E(K)$, given by $[S^1] \mapsto [S^1]$, $m_b \mapsto \mu_K$, and $\mu_T \mapsto \lambda_K$, provides the Finashin's annulus rim surgery. This operation obviously yields a band $b_K \subset B^3$ by knotting the band b along K and let f_K be the pair of arcs bounding b_K . Here the framing of b_K is chosen the same as the framing of b . So the resulting manifold of the annulus rim surgery performed on $\nu(M) \cong S^1 \times (B^3, f)$ becomes $S^1 \times (B^3, f_K)$ and we write a new pair as follows:

$$(8) \quad (X, \Sigma_K(\varphi)) = (X, \Sigma) - S^1 \times (B^3, f) \cup S^1 \times (B^3, f_K)$$

This construction can be further modified by twists along a meridian and a longitude of K , but we will stick to Finashin's construction; see [15] for other modifications.

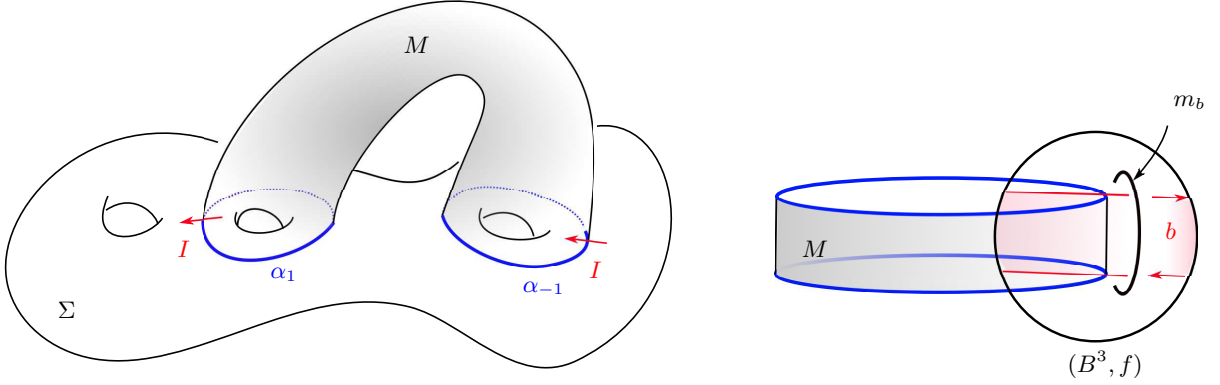


FIGURE 2. $T = S^1 \times m_b \subset S^1 \times (B^3, f) \cong \nu(M)$

Note that when $\pi_1(X - \Sigma) = \mathbb{Z}_d$, any (twisted or otherwise) annulus rim surgery preserves surface knot groups [6], [15, Proposition 3.3].

4. BASIC CONSTRUCTION

In order to get our main theorems, for two knots K and K' related by a single crossing change we will show the 1-stable equivalence on surface knots $\Sigma_K(\varphi)$ and $\Sigma_{K'}(\varphi)$ produced by knot surgery. The complete proof for each knotting construction will be given in Section 5.1, 5.2, 5.3 but in this section we will first discuss some constructions and properties that will be used repeatedly in the proofs of our stabilization results.

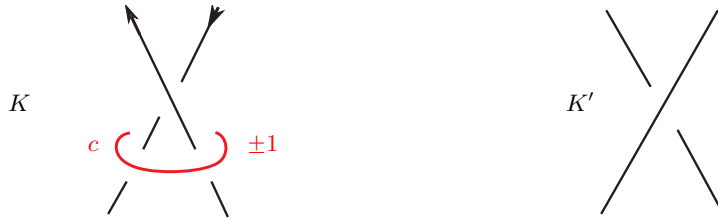


FIGURE 3. ± 1 -Dehn surgery along c at a crossing

Suppose that two knots K, K' in S^3 differ by a single crossing change, so that the knot K' is obtained by performing a ± 1 -Dehn surgery along a curve c around an oppositely oriented crossing of K as in Figure 3. Let $(X_K(\varphi), \Sigma_K(\varphi)), (X_{K'}(\varphi), \Sigma_{K'}(\varphi))$ be two pairs obtained by a knot surgery along a torus $T \subset X - \Sigma$ and gluing map φ along the knots K and K' respectively. Then we begin by showing that these pairs are related by a torus surgery:

Lemma 4.1. *A log transform of multiplicity ± 1 on $(X_K(\varphi), \Sigma_K(\varphi))$ produces the pair $(X_{K'}(\varphi), \Sigma_{K'}(\varphi))$.*

Proof. We first recall that the knot surgered pair is defined as follows:

$$(X_K(\varphi), \Sigma_K(\varphi)) = (X, \Sigma) - \nu(T) \cup_{\varphi} S^1 \times E(K).$$

Here, we denote T_c a torus $S^1 \times c$ in $S^1 \times E(K)$ of this decomposition, where c is a curve at a crossing of K as in Figure 3. We identify a neighborhood $\nu(T_c)$ with $S^1 \times (c \times D^2)$, where $c \times D^2$ is a neighborhood of c in $E(K)$. Then the ± 1 -log transform parallel to the curve c on T_c in $(X_K(\varphi), \Sigma_K(\varphi))$ can be described by the identity in the S^1 direction times the ± 1 -Dehn surgery along $c \subset E(K)$. Note that this process hasn't change the boundary $S^1 \times \partial E(K)$, so the resulting manifold is obtained by performing a torus surgery along T_c in $S^1 \times E(K)$, denoted by $(S^1 \times E(K))_{T_c}$, and gluing back in $(X, \Sigma) - \nu(T)$ along their boundaries via φ . This manifold is easily identified with $(X_{K'}(\varphi), \Sigma_{K'}(\varphi))$. It follows from that there is an obvious diffeomorphism from $(S^1 \times E(K))_{T_c}$ to $S^1 \times E(K')$ which carries each element in a basis $\{[S^1], \mu_K, \lambda_K\}$ of $H_1(\partial(S^1 \times E(K))_{T_c})$ to each element in $\{[S^1], \mu_{K'}, \lambda_{K'}\}$ of $H_1(S^1 \times \partial E(K'))$ respectively. \square

Remark 4.2. When the knot surgery $(X, \Sigma) \rightarrow (X, \Sigma_K(\varphi))$ is an ambient surgery such as (twisted) rim surgery and annulus rim surgery, the above torus surgery on $(X, \Sigma_K(\varphi))$ provides a new embedding $\Sigma_{K'}(\varphi)$ in X , which is given by the knot surgery performed on (X, Σ) along the same torus and gluing to produce $\Sigma_K(\varphi)$. And, it can be observed that $\pi_1(X_{K'}(\varphi) - \Sigma_{K'}(\varphi)) \cong \pi_1(X_K(\varphi) - \Sigma_K(\varphi))$ from the proof in Lemma 4.1, so if a knot surgery $(X, \Sigma) \rightarrow (X, \Sigma_K(\varphi))$ does not change the surface knot group $\pi_1(X - \Sigma)$ under some suitable circumstances then the new embedding $\Sigma_{K'}(\varphi)$ also preserves its knot group.

4.1. Fiber sum and gluing map. A torus surgery of $(X_K(\varphi), \Sigma_K(\varphi))$ along T_c can be described as a *fiber sum* of $(X_K(\varphi), \Sigma_K(\varphi))$ and $S^1 \times S^3$: Let T_u be a standardly embedded torus $S^1 \times u$ in $S^1 \times S^3$, where u is an unknot in S^3 . Then we write the torus surgered manifold in Lemma 4.1 as a fiber sum along tori T_c and T_u :

$$(9) \quad (X_{K'}(\varphi), \Sigma_{K'}(\varphi)) = (X_K(\varphi), \Sigma_K(\varphi))_{T_c} \#_{T_u} S^1 \times S^3.$$

In order to describe the gluing map $f : \partial\nu(T_u) \rightarrow \partial\nu(T_c)$ carefully, identify a tubular neighborhood $\nu(T_u)$ in $S^1 \times S^3$ with $S^1 \times (u \times D^2)$, where $u \times D^2$ is a neighborhood $\nu(u)$ of u in S^3 . Let $a = S^1 \times \{\text{pt}\} \subset S^1 \times u = T_u$ and a', u' be the pushoffs of a, u into $\partial\nu(T_u)$. Then $\{a', u', [\partial D^2]\}$ forms a basis for $H_1(\partial\nu(T_u))$. Similarly, under an identification $\nu(T_c) \cong S^1 \times (c \times D^2)$ in $S^1 \times E(K)$, let $\gamma = S^1 \times \{\text{pt}\} \subset S^1 \times c = T_c$ and $m_c = \{*\} \times (\{*\} \times \partial D^2)$, $l_c = \{*\} \times (c \times \{*\})$ be a meridian-longitude pair of c . So $\{\gamma', m_c, l_c\}$ gives a basis for $H_1(\partial\nu(T_c))$, where γ' is a pushoff of γ into $\partial\nu(T_c)$. As described in the proof of Lemma 4.1, the gluing map $f : S^1 \times (u \times \partial D^2) \rightarrow S^1 \times (c \times \partial D^2)$ is a diffeomorphism determined by

$$(10) \quad f_*(a') = \gamma', \quad f_*(u') = \pm m_c + l_c, \quad \text{and} \quad f_*([\partial D^2]) = \pm l_c.$$

In our present purpose, it is important to keep track of the gluing map in this fiber sum, from which we can determine the framing arising in our proof of the stabilization result.

4.2. Cobordism. In this section, we will construct a cobordism W whose upper boundary is $(X_K(\varphi), \Sigma_K(\varphi))_{T_c} \#_{T_u} S^1 \times S^3$ from $(X_K(\varphi), \Sigma_K(\varphi)) \sqcup S^1 \times S^3$. The proof of stable equivalence for surfaces $\Sigma_K(\varphi)$ and $\Sigma_{K'}(\varphi)$ will come from the middle level of the constructed cobordism W .

In terms of handles, we form W by adding 5-dimensional handles to $((X_K(\varphi), \Sigma_K(\varphi)) \sqcup S^1 \times S^3) \times I$ according to a standard handle decomposition of tori T_c and T_u , and their attaching maps will be determined from $f : \partial\nu(T_u) \rightarrow \partial\nu(T_c)$ in the fiber sum (10).

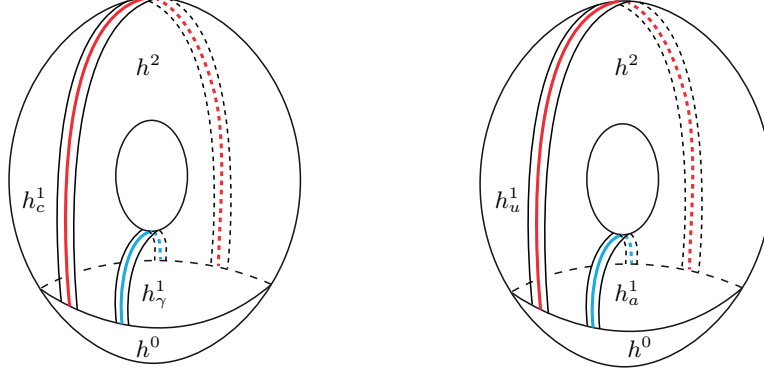


FIGURE 4. Handle decompositions of T_c and T_u

As in Figure 4 (thicken by D^2), a standard handle decomposition of $\nu(T_c) = T_c \times D^2$ can be given by one 0-handle h^0 , two 1-handles h^1_γ , h^1_c , and one 2-handle h^2 , where h^1_γ and h^1_c denote the 1-handles induced by the first and second factors of $T_c = S^1 \times c$ respectively. Similarly, $\nu(T_u) = T_u \times D^2$ has one 0-handle h^0 , two 1-handles h^1_a , h^1_u generated by the first and second factors of $T_u = S^1 \times u$, and one 2-handle h^2 . From the handles of these tori, W will be built by adding one 5-dimensional 1-handle H^1 , two 2-handles denoted by $H^2_{\gamma a}$, H^2_{cu} , and one 3-handle H^3 to $((X_K(\varphi), \Sigma_K(\varphi)) \sqcup S^1 \times S^3) \times I$.

To examine this attaching process closely, for each 4-dimensional k -handle $h^k = D^k \times D^{4-k}$ of a tubular neighborhood of a torus, define a 5-dimensional $(k+1)$ -handle H^{k+1} by $(D^k \times I) \times D^{4-k}$ where I denotes the interval $[-1, 1]$. Then the attaching map of a $(k+1)$ -handle H^{k+1} is described as follows. The discs $(D^k \times -1) \times 0$ and $(D^k \times 1) \times 0 \subset S^k = \partial(D^k \times I) \times 0$ are attached to each core of the k -handles of T_c and T_u respectively, and the rest $(\partial D^k \times I) \times 0$ connects the boundaries of these cores in $\partial_+ W_k$. The framing is completely determined by the gluing f in (10) and in particular, a 2-handle results in a surgery along a curve in the boundary.

While we're building a cobordism W from $(X_K(\varphi), \Sigma_K(\varphi)) \sqcup S^1 \times S^3$ to the fiber sum $(X_K(\varphi), \Sigma_K(\varphi))_{T_c \# T_u} S^1 \times S^3$, we will find out the resulting upper boundary at each stage of attaching handles. The level of W after a 1-handle H^1 is obviously $(X_K(\varphi), \Sigma_K(\varphi)) \# S^1 \times S^3$, and for the rest handles we will give more careful arguments.

Since we're interested in the equivalence of embeddings $\Sigma_K(\varphi)$ and $\Sigma_{K'}(\varphi)$ in a same manifold X , one may focus on ambient surgery so that $X_K(\varphi) \cong X$. Then all constructions of (twisted) rim surgery and annulus rim surgery will share the following diagram which indicates the boundary at each stage of adding handles:

$$(11) \quad \begin{aligned} (X, \Sigma_K(\varphi)) \sqcup S^1 \times S^3 &\xrightarrow{H^1} (X \# S^1 \times S^3, \Sigma_K(\varphi)) \xrightarrow{H^2_{\gamma a}} (X \# S^4, \Sigma_K(\varphi)) \xrightarrow{H^2_{cu}} \\ &\xrightarrow{H^3} (X \# S^2 \tilde{\times} S^2, \Sigma_K(\varphi)) \xrightarrow{H^3} (X, \Sigma_K(\varphi))_{T_c \# T_u} S^1 \times S^3 \cong (X, \Sigma_{K'}(\varphi)). \end{aligned}$$

Our main argument of the proof of the stabilization for $\Sigma_K(\varphi)$ and $\Sigma_{K'}(\varphi)$ follows from the middle level of W : Let W_k be a handlebody obtained by attaching all handles of index \leq

k to $((X, \Sigma_K(\varphi)) \sqcup S^1 \times S^3) \times I$. Then from the diagram (11), $\partial_+ W_2$ is $(X \# S^2 \tilde{\times} S^2, \Sigma_K(\varphi))$, which will be shown in Lemma 5.3, Theorem 5.5, and 5.6. After adding a 3-handle to W_2 , we would have a cobordism W from $(X, \Sigma_K(\varphi)) \sqcup S^1 \times S^3$ to the fiber sum which is diffeomorphic to X itself containing the surface $\Sigma_{K'}(\varphi)$ by Lemma 4.1 and (9). Turning the 3-handle H^3 upside down so that it becomes to attach a 2-handle H_*^2 to the fiber sum gives a connected sum with $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$ on X . Disregarding the surface $\Sigma_{K'}(\varphi)$ in the fiber sum, we first note that the level after attaching the 2-handle H_*^2 to X is same as $\partial_+ W_2$ which is diffeomorphic to $X \# S^2 \tilde{\times} S^2$. But since we're building a relative cobordism, it has to be argued that attaching the 2-handle H_*^2 gives rise to *the pair* $(X \# S^2 \tilde{\times} S^2, \Sigma_{K'}(\varphi))$ on the boundary. This will be verified in Lemma 5.4, Theorem 5.5, and 5.6 so that it will prove the 1-stable equivalence of $\Sigma_K(\varphi)$ and $\Sigma_{K'}(\varphi)$.

When we discuss about the stabilization in Section 6 for the case that knot surgery $(X, \Sigma) \rightarrow (X_K(\varphi), \Sigma_K(\varphi))$ is cyclic i.e. $\pi_1(X_K(\varphi) - \Sigma_K(\varphi)) \cong \pi_1(X - \Sigma)$ is cyclic, the cobordism W will be considered to be from $(X_K(\varphi), \Sigma_K(\varphi)) \sqcup S^1 \times S^3$ and it will be shown that the diagram (11) also works for this.

In the following subsections, we will investigate the level of W at each step of adding handles, and give some assertions that will be used in the proof of the stabilization for each knotting construction. For the purpose in this article, the knot surgery $(X, \Sigma) \rightarrow (X_K(\varphi), \Sigma_K(\varphi))$ is assumed to be cyclic or be an ambient surgery to produce a new surface $\Sigma_K(\varphi)$ in X throughout the rest of paper, although some proofs may work for more general cases.

4.2.1. Attaching a 2-handle $H_{\gamma a}^2$. Note that the homotopy class γa of attaching circle of 2-handle $H_{\gamma a}^2$ is represented by a curve $\gamma + a$ in the outer boundary of $((X_K(\varphi), \Sigma_K(\varphi)) \sqcup S^1 \times S^3) \times I \cup H^1$ as depicted in Figure 5. We claim that the resulting manifold on the boundary is $(X_K(\varphi) \# S^4, \Sigma_K(\varphi))$. This holds regardless what the homotopy class of γ is in $X_K(\varphi) - \Sigma_K(\varphi)$, but we can also know from the gluing map in each knot surgery construction that γ is in fact nullhomotopic in the case of rim surgery and annulus rim surgery, and is homotopically meridian of the surface $\Sigma_K(\varphi)$ for twist rim surgery under the hypothesis in Theorem C.

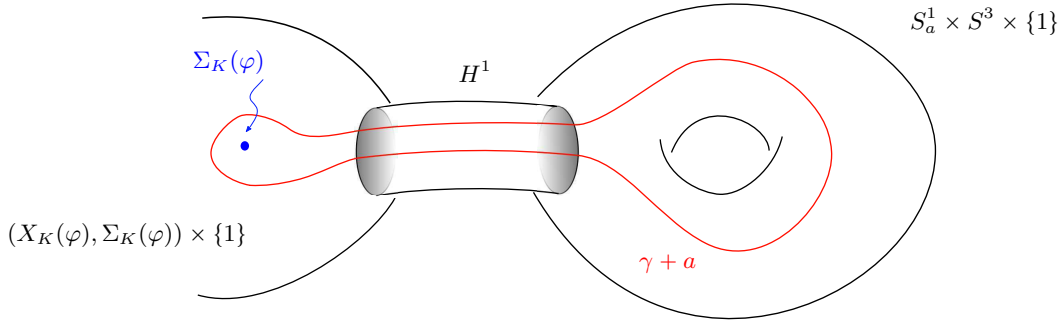


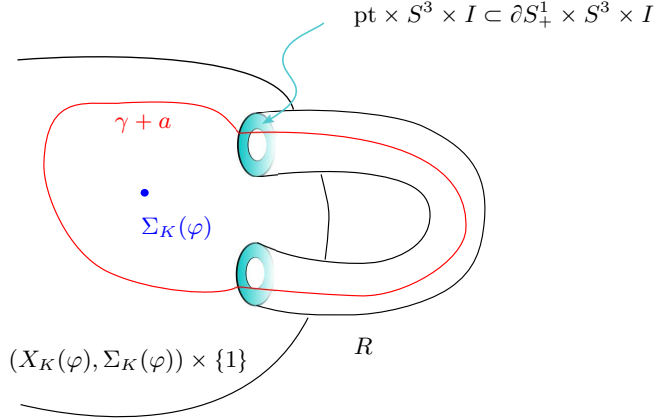
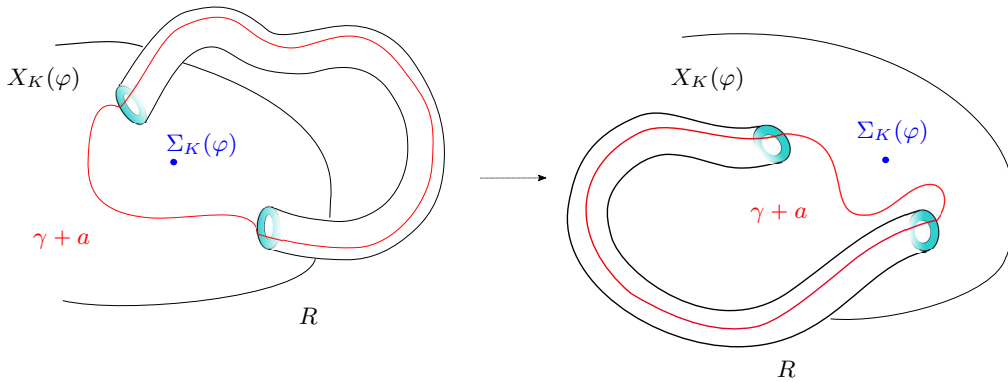
FIGURE 5. Attached 1-handle H^1 to $((X_K(\varphi), \Sigma_K(\varphi)) \sqcup S^1 \times S^3) \times \{1\}$

Lemma 4.3. *The resulting upper boundary of attaching a 2-handle $H_{\gamma a}^2$ to W_1 is diffeomorphic to $(X_K(\varphi) \# S^4, \Sigma_K(\varphi))$.*

Proof. Recall the curve $\gamma = S^1 \times \text{pt} \subset T_c = S^1 \times c \subset S^1 \times E(K)$ in the decomposition for $(X_K(\varphi), \Sigma_K(\varphi))$ and the curve $a = S^1 \times \text{pt} \subset T_u = S^1 \times u \subset S^1 \times S^3$. If γ is nullhomotopic in $X_K(\varphi) - \Sigma_K(\varphi)$ as in the case of rim surgery and annulus rim surgery, the result easily follows; attaching a 2-handle along $\gamma + a$ gives the effect on the boundary, that surgers out the curve $a = S^1 \times \text{pt} \subset X_K(\varphi) \# S^1 \times S^3$, so it produces $(X_K(\varphi) \# S^4, \Sigma_K(\varphi))$. It will be readily verified that the same surgery effect arises in any knot surgery $(X, \Sigma) \rightarrow (X_K(\varphi), \Sigma_K(\varphi))$.

We first draw $((X_K(\varphi), \Sigma_K(\varphi)) \sqcup S^1 \times S^3) \times I \cup H^1$ as Figure 6, which is basically obtained by attaching a ‘round handle’ $R := S^1_+ \times S^3 \times I$, where S^1_+ denotes a 1-handle of S^1 , to the outer boundary $(X_K(\varphi), \Sigma_K(\varphi)) \times \{1\}$ along $\partial S^1_+ \times S^3 \times I$. Then note that the attaching circle of the 2-handle $H^2_{\gamma+a}$ intersects the belt sphere of the 1-handle coming from the S^1 -factor in $S^1 \times S^3$ transversely at a single point in the outside of $\Sigma_K(\varphi) \times I$, so they are cancelled in $((X_K(\varphi) - \Sigma_K(\varphi)) \sqcup S^1 \times S^3) \times I \cup H^1$ to give the pair $(X_K(\varphi) \# S^4, \Sigma_K(\varphi))$ in the outer boundary.

One can also see that $\gamma + a$ is isotoped in $X_K(\varphi) \# S^1 \times S^3 - \Sigma_K(\varphi)$ to a as demonstrated in Figure 7 so that it yields a pairwise diffeomorphism $(X_K(\varphi) \# S^1 \times S^3, \Sigma_K(\varphi), \gamma + a) \rightarrow (X_K(\varphi) \# S^1 \times S^3, \Sigma_K(\varphi), a)$.


 FIGURE 6. $((X_K(\varphi), \Sigma_K(\varphi)) \sqcup S^1 \times S^3) \times I \cup H^1$

 FIGURE 7. Isotopy of round handle R

□

4.2.2. Attaching a 2-handle H_{cu}^2 and a dual handle H_*^2 of 3-handle H^3 . We now deal with the next 2-handle H_{cu}^2 and the dual 2-handle H_*^2 of 3-handle H^3 in building W . The level of our *relative* cobordism W after adding those handles will have more subtle issues and depend on the knotting construction, and so the details of the analysis for the boundary will be referred to the next following sections. But here we will focus on the ambient manifold to study the boundary after adding the handles without concerning surfaces.

As seen in Lemma 4.3, the level of W after adding a 2-handle $H_{\gamma a}^2$ is diffeomorphic to $X_K(\varphi) \# S^4 \cong X_K(\varphi)$. Note that if the knot surgery $(X, \Sigma) \rightarrow (X_K(\varphi), \Sigma_K(\varphi))$ is cyclic then $(X_K(\varphi), \Sigma_K(\varphi))$ is pairwise homeomorphic to (X, Σ) as shown in [15, Theorem 1.2], so the ambient manifold $X_K(\varphi)$ is simply-connected. When the knot surgery is cyclic or ambient surgery, adding another 2-handle H_{cu}^2 along the curve $c + u$ in $X_K(\varphi) \# S^4$ will give a connected sum with a S^2 -bundle over S^2 on $X_K(\varphi)$. The following elementary lemma determines the framing.

Lemma 4.4. *Attaching a 2-handle H_{cu}^2 to $X_K(\varphi)$ produces a connected sum with the twisted S^2 -bundle over S^2 so that $\partial_+ W_2$ is diffeomorphic to $X_K(\varphi) \# S^2 \tilde{\times} S^2$.*

Proof. Note that the surgery framing coming from attaching the 2-handle H_{cu}^2 along $c + u$ in $X_K(\varphi) \# S^4$ is determined by the gluing f in the fiber sum; $f_*(u') = \pm m_c + l_c$ in (10). Since c is nullhomotopic in $X_K(\varphi)$, it bounds a disk D_0 in $X_K(\varphi)$ that may intersect with the embedded surface $\Sigma_K(\varphi)$. Our framing along c relative to this disk is given by $\pm m_c + l_c$, so it does not extend over D_0 and gives $X_K(\varphi) \# S^2 \tilde{\times} S^2$. \square

Turning the cobordism W upside down, denoted by W^* , yields a dual 2-handle H_*^2 of the 3-handle which is attached to a collar $\partial_+ \overline{W} \times I = \partial_- W^* \times I = X_{K'}(\varphi) \times I$. Again this 2-handle H_*^2 will give a S^2 -bundle over S^2 on $X_{K'}(\varphi)$ as follows.

Lemma 4.5. *The level of W^* after adding a dual 2-handle H_*^2 is diffeomorphic to $X_{K'}(\varphi) \# S^2 \tilde{\times} S^2$.*

Proof. We first recall that the attaching sphere of the 3-handle H^3 is a sphere connecting the 2-handles $h_{T_u}^2$ of T_u and $h_{T_c}^2$ of T_c , and so the normal disk of T_u is a cocore of H^3 whose boundary will be glued with a curve in $\partial \nu(T_c)$ determined from f in the fiber sum $X_K(\varphi)_{T_c} \#_{T_u} S^1 \times S^3 \cong X_{K'}(\varphi)$. Note that $f_*([\partial D^2]) = \pm l_c$ in (10) which is a parallel curve of c , denoted by $\pm c'$, lying in the component $S^1 \times E(K)$ of $X_K(\varphi)$. This means that the attaching circle of the dual 2-handle H_*^2 in $X_{K'}(\varphi)$ is a curve c' at a crossing of the knot K' , which lies in the same position of the curve c at a crossing of K (up to isotopy). Regarding the framing, since $\partial_+(\partial_- W^* \times I \cup H_*^2) = \partial_+ W_2 \cong X_K(\varphi) \# S^2 \tilde{\times} S^2$ from Lemma 4.4, if X is spin then $\partial_+(\partial_- W^* \times I \cup H_*^2)$ must be diffeomorphic to $X_{K'}(\varphi) \# S^2 \tilde{\times} S^2$ (not $S^2 \times S^2$). The result easily follows when X is non-spin. \square

In the next following sections, for all known methods based on knot surgery constructing knotted surfaces we will verify the rest process in the digram (11) and prove that the surfaces $\Sigma_K(\varphi)$, $\Sigma_{K'}(\varphi)$ in X are 1-stably equivalent.

5. 1-STABLE EQUIVALENCE OF KNOTTED SURFACES

5.1. Twist rim surgery. Let $\pi_1(X - \Sigma)$ be any group G , and suppose that the surface $\Sigma \subset X$ carries a nontrivial homology class with $H_1(X - \Sigma) = \mathbb{Z}_d$. Then we state the following key theorem:

Theorem 5.1. *Suppose that two knots K, K' in S^3 differ by a single crossing change. If $(X, \Sigma_K(m))$ is a pair produced by an m -twist rim surgery such that either $m = \pm 1$ or the meridian μ_Σ has order d in $\pi_1(X - \Sigma)$ and $(m, d) = 1$, then $(X \# S^2 \tilde{\times} S^2, \Sigma_K(m))$ is pairwise diffeomorphic to $(X \# S^2 \tilde{\times} S^2, \Sigma_{K'}(m))$.*

To prove Theorem 5.1, we now turn to the 2-handle H_{cu}^2 and 3-handle in the diagram (11).

I. Attaching a 2-handle H_{cu}^2 . We have seen in Lemma 4.3 that the upper boundary after adding a 2-handle $H_{\gamma a}^2$ is diffeomorphic to $(X \# S^4, \Sigma_K(m))$, and from Lemma 4.4 adding another 2-handle H_{cu}^2 along the curve $c + u$ in $(X \# S^4, \Sigma_K(m))$ gives $X \# S^2 \tilde{\times} S^2$ for the ambient manifold. But since our stabilization is performed in the ‘outside’ of the surface $\Sigma_K(m)$, we first show that c is nullhomotopic in $X - \Sigma_K(m)$:

Proposition 5.2. *Suppose that $\Sigma \subset X$ is a surface carrying $H_1(X - \Sigma) = \mathbb{Z}/d$. If either $m = \pm 1$ or the meridian μ_Σ has order d in $\pi_1(X - \Sigma)$ and $(m, d) = 1$, then the curve c is nullhomotopic in $X - \Sigma_K(m)$.*

Proof. Since c is a curve at an oppositely oriented crossing of a knot K as in Figure 3, its homotopy class c can be expressed as $g^{-1}\mu_K^{-1}g\mu_K$ in terms of some element $g \in \pi_1(E(K))$. It easily follows from the presentation (7) for $\pi_1(X - \Sigma_K(m))$ in Proposition 3.1 that $c = g^{-1}\mu_K^{-1}g\mu_K = 1$ when either $m = \pm 1$ or $\mu_\Sigma^d = 1$ and $(m, d) = 1$. \square

Proposition 5.2 asserts that we can find a disk $D \subset X - \Sigma_K(m)$ such that $\partial D = c$, from which it follows that there is an induced diffeomorphism from any surgered manifold of X along c to the connected sum of X with a S^2 -bundle over S^2 i.e. $(X, \Sigma_K(m)) \# (S^2 \times S^2, \emptyset)$ or $(S^2 \tilde{\times} S^2, \emptyset)$. Note that $(X, \Sigma_K(m)) \# (S^2 \times S^2, \emptyset)$ is obtained from X by surgery on c with the framing determined by the unique normal framing of D , and $(X, \Sigma_K(m)) \# (S^2 \tilde{\times} S^2, \emptyset)$ is obtained with the other framing. To address the framing, we will explicitly find a disk and examine the framing of surgery relative to the disk.

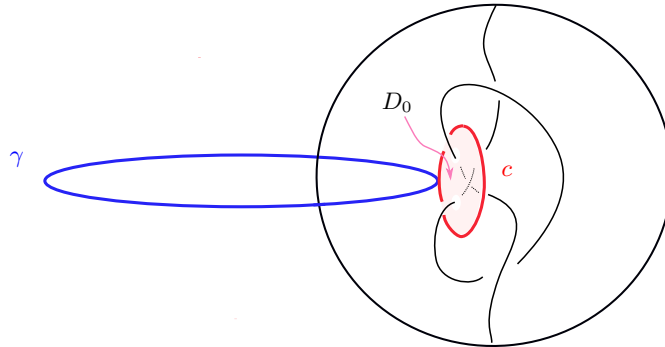


FIGURE 8. Torus $T_c = \gamma \times c \subset S^1 \times_{\tau^m} (B^3, K_+)$

We first recall that twist rim surgery is performed along a rim torus in $\nu(\alpha) \cong S^1 \times (B^3, I)$ so that it produces the mapping torus $S^1 \times_{\tau^m} (B^3, K_+)$; see (3). So we view the torus $T_c = \gamma \times c = S^1 \times c \subset S^1 \times E(K)$ used in a log transform in Lemma 4.1 is lying in the mapping torus $S^1 \times_{\tau^m} (B^3, K_+)$; see Figure 8, and so the curve c is in $S^1 \times_{\tau^m} (B^3 - K_+)$.

Lemma 5.3. *The upper boundary of 2-handlebody W_2 in the cobordism W is diffeomorphic to $(X \# S^2 \tilde{\times} S^2, \Sigma_K(m))$.*

Proof. Attaching a 2-handle H_{cu}^2 gives a surgery along c in $(X, \Sigma_K(m)) \cong (X \# S^4, \Sigma_K(m))$. This curve c obviously bounds an embedded disk D_0 in the component $S^1 \times_{\tau^m} (B^3, K_+)$ of the decomposition (3) for $(X, \Sigma_K(m))$ that intersects with $\Sigma_K(m)$ at two points as in Figure 8. Using the disk D_0 , we denote by ψ_0 the surgery framing induced from attaching the 2-handle H_{cu}^2 . Note that there is a framing determined by the unique normal framing of D_0 , but by Lemma 4.4, our framing ψ_0 on $c = \partial D_0$ relative to this disk is the other one so that it doesn't extend over D_0 . To investigate the framing on c in $X - \Sigma_K(m)$, we shall find a disk D bounding c in the complement of $\Sigma_K(m)$, and compare ψ_0 with the framing ψ on c determined by the disk D . This can be checked by computing $\langle w_2(X), [D_0 \cup -D] \rangle$ since $\psi_0 \equiv \psi \pmod{2} \Leftrightarrow \langle w_2(X), [D_0 \cup -D] \rangle = 0$.

To find such a disk D , we shall chase the homotopy class c in the presentation for $\pi_1(X - \Sigma_K(m))$ given in Proposition 3.1. Referring to the decomposition (4) for $X - \Sigma_K(m)$, we first claim that c bounds a punctured torus in $S^1 \times_{\tau^m} (B^3 - K_+)$. The twist map τ along a meridian of K gives a relation $\tau_*^m(g) = \mu_K^{-m} g \mu_K^m$ for all $g \in \pi_1(B^3 - K_+)$, which is same as $\mu_K^{-1} g \mu_K$ under the assumption on m that is given by either $m = \pm 1$ or $(m, d) = 1$ and $\mu_\Sigma^d = 1$. And since the homotopy class c is $g^{-1} \mu_K^{-1} g \mu_K$ for some $g \in \pi_1(B^3 - K_+)$, it is same as $g^{-1} \tau_*^m(g)$, which obviously bounds a punctured torus in $S^1 \times_{\tau^m} (B^3 - K_+)$; see the first picture in Figure 9.

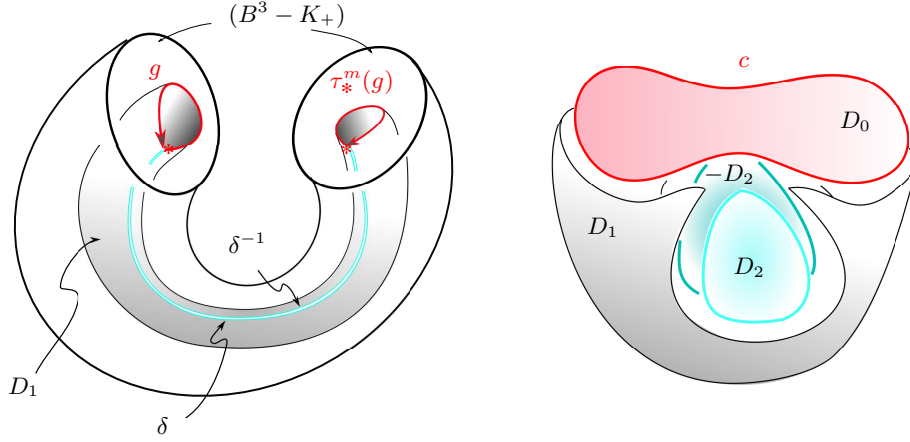


FIGURE 9. Disk D in $X - \Sigma_K(m)$ with $\partial D = c$

Now consider a relation $\delta^{-1}g\delta = \tau_*^m(g)$ in $\pi_1(S^1 \times_{\tau^m} (B^3 - K_+))$ where δ denotes a generator $[S^1]$ of $\pi_1(S^1 \times_{\tau^m} (B^3 - K_+), *)$ in Figure 1, so a curve representing $\delta^{-1}g^{-1}\delta\tau_*^m(g)$ bounds a disk D_1 as in the first picture of Figure 9. And, δ is nullhomotopic in $X - \Sigma_K(m)$ from the presentation (6) in Proposition 3.1, so it bounds a disk D_2 in $X - \Sigma - S^1 \times (B^3, I)$. Adding this relation to the presentation of c , we write $c = \delta^{-1}g^{-1}\delta\tau_*^m(g)$, which bounds a disk $D = D_1 \cup D_2 \cup -D_2$ in $X - \Sigma_K(m)$ as depicted in the second picture of Figure 9.

It remains to compare the framings ψ_0, ψ . Note that the element $[D_0 - D] \in H_2(X)$ is same as $[D_0 - D_1] \in H_2(S^1 \times_{\tau^m} (B^3, K_+))$ represented by a torus; see the second picture in Figure 9, which is trivial in $H_2(S^1 \times_{\tau^m} (B^3, K_+)) = 0$, so does in $H_2(X)$. This shows $w_2(X)$ vanishes on this class, from which our result follows. \square

II. Attaching a dual handle H_*^2 of 3-handle H^3 . We now turn to the 3-handle. Turning W upside down, the 3-handle provides adding a 2-handle H_*^2 to $\partial_+ \overline{W} = X_{T_c \# T_u} S^1 \times S^3 \cong (X, \Sigma_{K'}(m))$ in our relative cobordism. As it turns out in Lemma 4.5, a key point is that

the dual handle H_*^2 is attached to a curve c' at a crossing of K' and its framing, disregarding surface knot $\Sigma_{K'}(m)$ in X , is shown to be twisted.

This means that the surgery framing from adding H_*^2 can be specified by some disk $D_0 \subset X$ with $\partial D_0 = c'$ that may intersect with $\Sigma_{K'}(m)$, and the framing cannot extend over the full disk D_0 . But since we're building a relative cobordism from $X_{T_c \# T_u} S^1 \times S^3 \cong (X, \Sigma_{K'}(m))$, we will find a disk D in $X - \Sigma_{K'}(m)$ bounding c' , and then examine the surgery framing relative to this disk D . The idea is same as before, so it is basically to find a dual sphere of the attaching sphere of H^3 that doesn't intersect with $\Sigma_{K'}(m)$ and determine its framing.

Lemma 5.4. *The upper boundary of $\partial_- W^* \times I \cup H_*^2$ is diffeomorphic to $(X \# S^2 \tilde{\times} S^2, \Sigma_{K'}(m))$.*

Proof. Let's first consider the attaching region of the 3-handle $H^3 = D^3 \times D^2 = (D^2 \times I) \times D^2$. Described in Section 4.2, the parts $(D^2 \times -1) \times 0$ and $(D^2 \times 1) \times 0$ of the attaching sphere $S^2 = \partial(D^2 \times I) \times 0$ of H^3 are attached to each 2-handle $h_{T_c}^2$, $h_{T_u}^2$ of T_c and T_u respectively; see the black sphere in Figure 10.

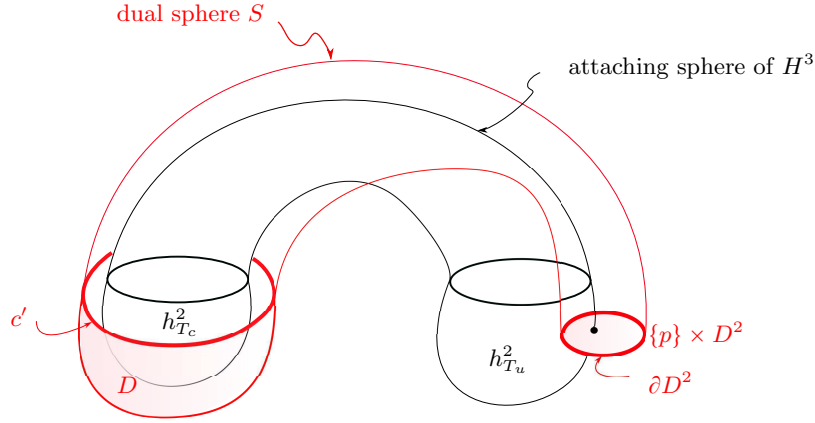


FIGURE 10. Attaching sphere of H^3 and its dual sphere S

In order to find its dual sphere, we start with a parallel of the cocore of the 3-handle $\{p\} \times D^2 \subset \partial(D^2 \times I) \times D^2$ where $p \in (D^2 \times 1) \times 0$. This is a normal disk of the attaching sphere of H^3 ; see the red disk intersecting with $h_{T_u}^2$ at one point in Figure 10. From the proof in Lemma 4.5, note that the boundary $\{p\} \times \partial D^2$ of the normal disk of T_u is glued to a curve c' in $X_{T_c \# T_u} S^1 \times S^3 \cong (X, \Sigma_{K'}(m))$ so that H_*^2 is attached along a curve c' at a crossing of K' . Then under our current assumption on m , Proposition 5.2 shows that c' is nullhomotopic in $X - \Sigma_{K'}(m)$, so this surgery gives $(S^2 \times S^2, \emptyset)$ or $(S^2 \times S^2, \emptyset)$ on $(X, \Sigma_{K'}(m))$.

To decide the framing, note that the curve c' is in $S^1 \times E(K') \subset (X, \Sigma_{K'}(m))$ as well as in the surgered manifold $S^1 \times_{\tau^m} (B^3, K'_+)$ of the neighborhood $\nu(\alpha) \cong S^1 \times (B^3, I)$. So, the curve c' bounds a disk D_0 in $S^1 \times_{\tau^m} (B^3, K'_+)$ intersecting with $\Sigma_{K'}(m)$ at two points as c does in Figure 8. The hypothesis on m in $\Sigma_{K'}(m)$ allows one to find a disk D in $X - \Sigma_{K'}(m)$ with the exactly same argument in Lemma 5.3, which gives a dual sphere of H^3 ; see a red sphere drawn in Figure 10. Repeating the argument in Lemma 5.3 shows that the framing on c' relative to D is equivalent to the one relative to D_0 up to (mod 2) that does not extend over the full disk D_0 by Lemma 4.5. \square

As shown in Lemma 5.3 and Lemma 5.4, we have $(X \# S^2 \tilde{\times} S^2, \Sigma_K(m)) \cong \partial_+ W_2 = \partial_+(\partial_- W^* \times I \cup H_*^2) \cong (X \# S^2 \tilde{\times} S^2, \Sigma_{K'}(m))$, so it completes the proof of Theorem 5.1.

5.2. Rim surgery. Finshel-Stern's rim surgery is the case $m = 0$ of m -twist rim surgery. Let Σ be an embedded surface in a simply-connected 4-manifold X with $\pi_1(X - \Sigma) = 1$. As discussed in Section 3.1, rim surgery performed in a neighborhood $\nu(\alpha) \cong S^1 \times (B^3, I)$ of a curve $\alpha \subset \Sigma$ produces $S^1 \times (B^3, K_+)$ i.e. $m = 0$ in (3).

Theorem 5.5. *Suppose that two knots K, K' in S^3 differ by a single crossing change. If $\Sigma_K(\varphi)$ and $\Sigma_{K'}(\varphi)$ are surface knots obtained by rim surgery then $(X \# S^2 \tilde{\times} S^2, \Sigma_K(\varphi))$ is pairwise diffeomorphic to $(X \# S^2 \tilde{\times} S^2, \Sigma_{K'}(\varphi))$.*

Proof. Lemma 4.3 asserts that the level of W after adding a 1-handle H^1 and a 2-handle $H_{\gamma a}^2$ is $(X \# S^4, \Sigma_K(\varphi))$. At the stage of adding the next 2-handle H_{cu}^2 along $c + u$, since $\pi_1(X - \Sigma_K(\varphi)) = 1$, the curve c is nullhomotopic in $X - \Sigma_K(\varphi)$, so the surgery from the 2-handle gives a connected sum of $(S^2 \times S^2, \emptyset)$ or $(S^2 \tilde{\times} S^2, \emptyset)$. We now need to handle with the framing issue.

Since c bounds a disk D_0 in $S^1 \times (B^3, K_+) \subset (X, \Sigma_K(\varphi))$ intersecting at two points with $\Sigma_K(\varphi)$ as in Figure 8, this disk specifies the framing ψ_0 on the curve c induced from the 2-handle H_{cu}^2 , which doesn't extend over D_0 from Lemma 4.4. To find a disk D with $\partial D = c$ in $X - \Sigma_K(\varphi)$, note that c is at a crossing of the knot K , so it bounds an obvious punctured torus T_* in $E(K)$ consisting of two generators; a meridian μ_K of K and some $g \in \pi_1(E(K))$ represented by a blue curve as in Figure 11. Since $\pi_1(X - \Sigma_K(\varphi))$ is trivial, the image of $\pi_1(S^1 \times E(K))$ is trivial so that the curve g bounds a disk D_2 in $X - \Sigma - \nu(R_\alpha)$ where R_α denotes the rim torus given by a curve $\alpha \subset \Sigma$. Cutting T_* along g and filling with two oppositely oriented disks $D_2 \cup -D_2$ gives a disk D bounding c in $X - \Sigma_K(\varphi)$.

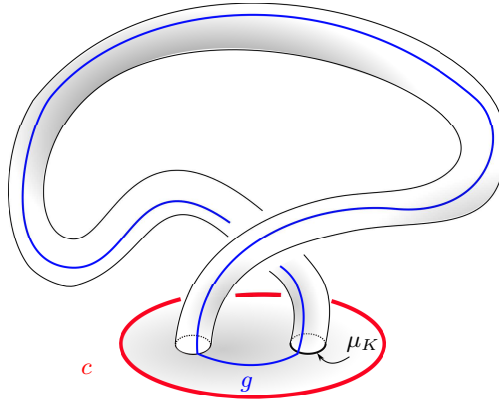


FIGURE 11. Punctured torus T_* in the exterior $E(K)$ of the knot K

If ψ denotes the framing on c relative to the disk D , it readily follows that ψ is equivalent to $\psi_0 \pmod{2}$ by showing $\langle w_2(X), [D_0 \cup -D] \rangle = 0$. This is because $[D_0 \cup -D] = [D_0 \cup T_*]$ is represented by a torus in $S^1 \times (B^3, K_+)$, which vanishes in $H_2(S^1 \times (B^3, K_+)) = 0$, so does in $H_2(X)$. This shows that the level $\partial_+ W_2$ is diffeomorphic to $(X \# S^2 \tilde{\times} S^2, \Sigma_K(\varphi))$.

Now turn W upside down and note that as shown in Lemma 5.4, the dual 2-handle H_2^* of the 3-handle H^3 has the attaching circle as a curve c' in $S^1 \times E(K')$, which is nullhomotopic in $X - \Sigma_{K'}(\varphi)$ since $\pi_1(X - \Sigma_{K'}(\varphi)) = 1$. The 2-handle H_2^* contributes a factor $(S^2 \times S^2, \emptyset)$ or $(S^2 \tilde{\times} S^2, \emptyset)$ to $(X, \Sigma_{K'}(\varphi))$. Repeating the same argument in the

above with c' in $(X, \Sigma_{K'}(\varphi))$, we can show that the framing of surgery along c' is also twisted outside the surface $\Sigma_{K'}(\varphi)$. So, we will have $(X \# S^2 \tilde{\times} S^2, \Sigma_{K'}(\varphi)) \cong \partial_+(\partial_- W^* \times I \cup H_*^2) = \partial_+ W_2 \cong (X \# S^2 \tilde{\times} S^2, \Sigma_K(\varphi))$. \square

5.3. Annulus rim surgery. Our setting is given as in Section 3.3, and recall that the Finashin's construction is a knot surgery along a torus in a neighborhood $\nu(M) \cong S^1 \times (B^3, f)$ to produce $S^1 \times (B^3, f_K) = S^1 \times (B^3, f) - S^1 \times (m_b \times D^2) \cup_\varphi S^1 \times E(K)$ with the gluing $[S^1] \mapsto [S^1]$, $m_b \mapsto \mu_K$, and $\mu_T \mapsto \lambda_K$. Furthermore, it is not hard to see that $\pi_1(X - \Sigma_K(\varphi))$ is preserved when $\pi_1(X - \Sigma) = \mathbb{Z}_d$ by applying the Van Kampen theorem for the decomposition (8) of $X - \Sigma_K(\varphi)$. In this computation, we see that the generators $[S^1]$, μ_K of $\pi_1(S^1 \times E(K))$ are trivial in $\pi_1(X - \Sigma_K(\varphi))$, so the image of $\pi_1(S^1 \times E(K))$ is a trivial subgroup of $\pi_1(X - \Sigma_K(\varphi))$; see [6], [15, Proposition 3.3] for more details. In this circumstance, the same argument in the rim surgery case works here.

Theorem 5.6. *Suppose that two knots K, K' in S^3 differ by a single crossing change. If $\Sigma_K(\varphi)$ and $\Sigma_{K'}(\varphi)$ are surface knots obtained by annulus rim surgery then $(X, \Sigma_K(\varphi)) \# (S^2 \tilde{\times} S^2, \emptyset)$ is pairwise diffeomorphic to $(X, \Sigma_{K'}(\varphi)) \# (S^2 \tilde{\times} S^2, \emptyset)$.*

Proof. We just begin with the 2-handle H_{cu}^2 attached along $c + u$ in $(X \# S^4, \Sigma_K(\varphi))$ from Lemma 4.3. Since c is a curve at a crossing of K and the annulus rim surgery is performed on a neighborhood $\nu(M)$, the curve c lies in the resulting manifold $S^1 \times (B^3, f_K) = S^1 \times (B^3, f) - S^1 \times (m_b \times D^2) \cup_\varphi S^1 \times E(K)$. And, it is nullhomotopic in $X - \Sigma_K(\varphi)$ since the image $\pi_1(S^1 \times E(K))$ is trivial in $\pi_1(X - \Sigma_K(\varphi))$ as shown in [6], [15, Proposition 3.3]. So we sketch the exactly same argument in Theorem 5.5.

There exists a disk D_0 bounding c in $S^1 \times (B^3, f_K)$ that intersects with $\Sigma_K(\varphi)$ at 'four points', and the surgery framing relative to the disk D_0 , coming from the 2-handle H_{cu}^2 , doesn't extend over D_0 by Lemma 4.4. Since c bounds a punctured torus T_* in $E(K)$ and the image $\pi_1(S^1 \times E(K))$ is trivial in $\pi_1(X - \Sigma_{K'}(\varphi))$, the argument in Theorem 5.5 gives a way to find another disk D in $X - \Sigma_K(\varphi)$ bounding c . It readily follows that the framing on c relative to D is also twisted since the homology class $[D_0 \cup -D]$ is represented by a torus $[D_0 \cup T_*]$ in $H_2(S^1 \times (B^3, f_K)) = 0$. So $w_2(X)$ vanishes on this class, from which we have $\partial_+ W_2 \cong (X \# S^2 \tilde{\times} S^2, \Sigma_K(\varphi))$.

Finally, turn W upside down. By Lemma 4.5, the dual 2-handle H_2^* of the 3-handle is attached along a curve c' in $S^1 \times E(K')$, which has a trivial π_1 in $\pi_1(X - \Sigma_{K'}(\varphi))$ so that the attaching circle c' is nullhomotopic in $X - \Sigma_{K'}(\varphi)$. One simply proceeds the above argument to show that the boundary $\partial_+(\partial_- W^* \times I \cup H_*^2)$ is diffeomorphic to $(X \# S^2 \tilde{\times} S^2, \Sigma_{K'}(\varphi))$, and hence our result follows. \square

Remark 5.7. A crucial point of our argument for the case of rim surgery and annulus rim surgery is that the image $\pi_1(S^1 \times E(K))$ is trivial in $\pi_1(X - \Sigma_{K'}(\varphi))$. This allows us to exhibit a disk bounding c easily as we first find a punctured torus T_* with $\partial T_* = c$ in $E(K)$ and surger out one of two generators of T_* using a disk bounding the circle. But the difference in twist rim surgery is that one cannot proceed this argument, and so we enlarge $S^1 \times E(K)$ to the surgered manifold $S^1 \times_{\tau^m} (B^3, K_+)$ of a neighborhood of a curve on Σ and use the fact that the image of $\pi_1(S^1 \times_{\tau^m} (B^3 - K_+))$ in $\pi_1(X - \Sigma_K(m))$ is a cyclic subgroup generated by the meridian of $\Sigma_K(m)$ under our hypothesis in Proposition 5.2.

5.4. Stabilization for Rim surgery, Annulus rim surgery, and Twist rim surgery; Proofs of Theorem A, B, and C.

Proof of Theorem A, B, and C. Suppose that $(X, \Sigma_K(\varphi))$ is a pair constructed from rim surgery, twisted rim surgery, and annulus rim surgery on (X, Σ) , and assume that it preserves its surface knot group under the given hypothesis of Theorems A, B, C. For any knot K in S^3 , there is a sequence of knots $K_1 = K, K_2, \dots, K_n$, with the unknot K_n , by crossing changes i.e. ± 1 -Dehn surgery along disjoint n -curves $\{c_i\}_{i=1, \dots, n}$ in $E(K)$. So, for each i the pair $(X, \Sigma_{K_{i+1}}(\varphi))$ is obtained by a (± 1) -log transform along a torus T_{c_i} in $(X, \Sigma_{K_i}(\varphi))$. And at each stage, the surface knot group $\pi_1(X - \Sigma_{K_i}(\varphi))$ is preserved for each knotting construction so that Theorem 5.1, 5.5, and 5.6 assert that $(X \# S^2 \tilde{\times} S^2, \Sigma_{K_i}(\varphi))$ is pairwise diffeomorphic to $(X \# S^2 \tilde{\times} S^2, \Sigma_{K_{i+1}}(\varphi))$, and hence we deduce that $(X \# S^2 \tilde{\times} S^2, \Sigma_K(\varphi))$ is pairwise diffeomorphic to $(X \# S^2 \tilde{\times} S^2, \Sigma_{K_n}(\varphi))$ where K_n is unknot.

In [15, Lemma 2.2], it is shown that for the unknot K_n , any knot surgery on $X - \nu(\Sigma)$ along a torus $T \subset X - \nu(\Sigma)$ and a gluing φ with $\varphi(\mu_T) = \lambda_{K_n}$ gives a diffeomorphism $(X - \nu(\Sigma))_{K_n} \rightarrow X - \nu(\Sigma)$ that is the identity on the boundary. Thus, $(X, \Sigma_{K_n}(\varphi)) \cong (X, \Sigma)$ so that this proves our main theorems. \square

6. STABILIZATION FOR CYCLIC KNOT SURGERY

Proof of Theorem E. We shall follow the standard argument for the 1-stable equivalence in the previous knotting constructions, so the first step is to show that for any two knots K and K' related by a single crossing change, the cyclic knot surgered pairs $(X_K(\varphi), \Sigma_K(\varphi))$ and $(X_{K'}(\varphi), \Sigma_{K'}(\varphi))$ become pairwise diffeomorphic after connected summing with $(S^2 \tilde{\times} S^2, \emptyset)$.

We work with the cobordism W constructed in Section 4.2, and the level of W after adding a 1-handle H^1 and a 2-handle $H_{\gamma_a}^2$ is diffeomorphic to $(X_K(\varphi) \# S^4, \Sigma_K(\varphi))$ by Lemma 4.3, so we just need to argue with a 2-handle H_{cu} and a 3-handle H^3 .

For the 2-handle H_{cu} , we first recall from Lemma 4.4 that attaching the 2-handle along a curve c in $(X_K(\varphi), \Sigma_K(\varphi))$ gives rise to $X_K(\varphi) \# S^2 \tilde{\times} S^2$ for the ambient manifold and so its framing on c is determined by a disk D_0 in $X_K(\varphi)$ that may intersect with $\Sigma_K(\varphi)$ and the framing does not extend over D_0 . To consider the framing in the complement of surface knot, we note that c is also nullhomotopic in $X_K(\varphi) - \Sigma_K(\varphi)$ because the element c is $g^{-1} \mu_K^{-1} g \mu_K$ for some $g \in \pi_1(E(K))$ and $\pi_1(X_K(\varphi) - \Sigma_K(\varphi))$ is cyclic. This means that there is a disk D in $X_K(\varphi) - \Sigma_K(\varphi)$ bounding c , and the framing of surgery along c relative to D can be compared with the one relative to the disk D_0 by evaluating $w_2(X_K(\varphi))$ on the element $[D_0 \cup -D]$, which turns out to be zero because $X_K(\varphi)$ is spin. Thus, the level $\partial_+ W_2$ is diffeomorphic to the pair $(X_K(\varphi) \# S^2 \tilde{\times} S^2, \Sigma_K(\varphi))$.

Turning the 3-handle upside down, it was shown in Lemma 4.5 that its dual 2-handle gives a surgery along c' on $(X_{K'}(\varphi), \Sigma_{K'}(\varphi))$, where the curve c' is at an oppositely oriented crossing of K' . Since $\pi_1(X_{K'}(\varphi) - \Sigma_{K'}(\varphi))$ is cyclic, c' is nullhomotopic in $X_{K'}(\varphi) - \Sigma_{K'}(\varphi)$ so that the surgery from the dual 2-handle yields a connected sum of $(S^2 \times S^2, \emptyset)$ or $(S^2 \tilde{\times} S^2, \emptyset)$ on the boundary. In the middle level of W between 2-handles and a 3-handle, we will have a pairwise diffeomorphism $(X_K(\varphi) \# S^2 \tilde{\times} S^2, \Sigma_K(\varphi)) \rightarrow (X_{K'}(\varphi) \# S^2 \times S^2, \Sigma_{K'}(\varphi))$ or $(X_{K'}(\varphi) \# S^2 \tilde{\times} S^2, \Sigma_{K'}(\varphi))$. But it must be the pair $(X_{K'}(\varphi) \# S^2 \tilde{\times} S^2, \Sigma_{K'}(\varphi))$ because both ambient manifolds $X_K(\varphi)$ and $X_{K'}(\varphi)$ are spin.

For the rest argument, the proof in Section 5.4 applies for the case of cyclic knot surgery with no extra effort. \square

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REFERENCES

- [1] Selman Akbulut, *Variations on Fintushel-Stern knot surgery on 4-manifolds*, Proceedings of 8th Gökova Geometry-Topology Conference, Gökova Geometry/Topology Conference (GGT), Gökova, 2002, pp. 1–12.
- [2] Dave Auckly, *Families of four-dimensional manifolds that become mutually diffeomorphic after one stabilization*, Top. Appl. **127** (2003), 277–298.
- [3] Dave Auckly, Hee Jung Kim, Paul Melvin, and Daniel Ruberman, *Isotopy of surfaces and diffeomorphisms after stabilization*, J. London Math. Soc. **91** (2015), 439–463.
- [4] R. İnanc Baykur and Nathan Sunukjian, *Round handles, logarithmic transforms and smooth 4-manifolds*, J. Topol. **6** (2013), no. 1, 49–63.
- [5] ———, *Knotted surfaces in 4-manifolds and stabilizations*, J. Topol. **9** (2016), no. 1, 215–231.
- [6] Sergey Finashin, *Knotted surfaces in \mathbb{CP}^2* , Topology. **41** (2002), no. 1, 47–55.
- [7] Sergey Finashin, M. Kreck, and O. YA. Viro, *Exotic knotting of surfaces in the 4-sphere*, Bulletin of American Mathematical Society. vol**41** (2002), no. 1, 47–55.
- [8] Ronald Fintushel and Ronald J. Stern, *Knots, links, and 4-manifolds*, Invent. Math. **134** (1998), 363–400.
- [9] Ronald Fintushel and Ronald J. Stern, *Surfaces in 4-manifolds*, Math. Res. Lett. **4** (1997), no. 6, 907–914.
- [10] ———, *Surfaces in 4-manifolds: Addendum*, arXiv:math/0511707.pdf (2005).
- [11] R. H. Fox, *Rolling*, Bull. Amer. Math. Soc. **72** (1966), 162–164.
- [12] Michael H. Freedman and Frank Quinn, *Topology of 4-manifolds*, Princeton University Press, Princeton, N.J., 1990.
- [13] R. E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. (2) **142** (1995), 527–595.
- [14] Hee Jung Kim, *Modifying surfaces in 4-manifolds by twist spinning*, Geom. Topol. **10** (2006), 27–56 (electronic).
- [15] Hee Jung Kim and Daniel Ruberman, *Topological triviality of smoothly knotted surfaces in 4-manifolds*, Trans. Amer. Math. Soc. **360** (2008), 5869–5881.
- [16] ———, *Smooth Surfaces with Non-simply-connected Complements*, Algebraic & Geometric Topology **8** (2008), 2263–2287.
- [17] ———, *Double point surgery and configurations of surfaces*, Journal of Topology **4** (3) (2011), 573–590.
- [18] R. A. Litherland, *Deforming twist-spun knots*, Trans. Amer. Math. Soc. **250** (1979), 311–331.
- [19] T. Mark, *Knotted surfaces in 4-manifolds*, Forum Math. **25** (2013), no. 3, 597–637.
- [20] Bernard Perron, *Pseudo-isotopies et isotopies en dimension quatre dans la catégorie topologique*, Topology **25** (1986), 381–397.
- [21] Frank Quinn, *The stable topology of 4-manifolds*, Topology and its Applications **15** (1983), 71–77.
- [22] ———, *Isotopy of 4-manifolds*, J. Differential Geom. **24** (1986), no. 3, 343–372.
- [23] Daniel Ruberman, *An obstruction to smooth isotopy in dimension 4*, Math. Res. Lett. **5** (1998), no. 6, 743–758.
- [24] ———, *A polynomial invariant of diffeomorphisms of 4-manifolds*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), Geom. Topol., Coventry, 1999, pp. 473–488 (electronic).
- [25] Nathan S. Sunukjian, *A note on knot surgery*, J. Knot Theory Ramifications **24** (2015), no. 9, 1520003, 5 pp.
- [26] C. T. C. Wall, *Diffeomorphisms of 4-manifolds*, J. London Math. Soc. **39** (1964), 131–140.
- [27] ———, *On simply-connected 4-manifolds*, J. London Math. Soc. **39** (1964), 141–149.
- [28] E. C. Zeeman, *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495.

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